

## Q-systems as cluster algebras

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## **$Q$ -systems as cluster algebras**

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### **Abstract**

$Q$ -systems first appeared in the analysis of the Bethe equations for the XXX model and generalized Heisenberg spin chains (Kirillov and Reshetikhin 1987 *Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst. Steklov.* **160** 211–21, 301). Such systems are known to exist for any simple Lie algebra and many other Kac–Moody algebras. We formulate the  $Q$ -system associated with any simple, simply-laced Lie algebras  $\mathfrak{g}$  in the language of cluster algebras (Fomin and Zelevinsky 2002 *J. Am. Math. Soc.* **15** 497–529), and discuss the relation of the polynomiality property of the solutions of the  $Q$ -system in the initial variables, which follows from the representation–theoretical interpretation, to the Laurent phenomenon in cluster algebras (Fomin and Zelevinsky 2002 *Adv. Appl. Math.* **28** 119–44).

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(Some figures in this article are in colour only in the electronic version)

### **1. Introduction**

For any simple Lie algebra, and for many other Kac–Moody algebras, there exists an associated system of equations known as a  $Q$ -system. These equations can be regarded as a recursion relation in  $i$  for the variables  $\{Q_{\alpha,i} : \alpha \in I_r, i \in \mathbb{Z}_+\}$ ,  $I_r = \{1, \dots, r\}$  where  $r$  is the rank of the algebra  $\mathfrak{g}$ .

Solutions to the  $Q$ -system describe characters of special representations of the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  or of the Yangian  $Y(\mathfrak{g})$ , as well as the classical limit described by Chari [2].

The  $Q$ -systems can be considered as a type of discrete dynamical system, and for the root system of type  $A$ , it is a specialization of the discrete Hirota equations. The  $Q$ -system first appeared in the analysis of the Bethe ansatz of the generalized inhomogeneous Heisenberg spin chain [12] in the thermodynamic limit. This system is instrumental to the solution of the

Kirillov–Reshetikhin (KR) conjectures about completeness of the Bethe ansatz solutions and about the structure of finite-dimensional representations of the quantum algebra [11, 19].

The  $Q$ -systems have the remarkable property that their solutions are polynomials, given appropriate initial data. This fact was the essential ingredient in the recent proof [3] of the combinatorial Kirillov–Reshetikhin conjectures, which are the completeness conjectures for the generalized Heisenberg spin chains.

On the other hand, the concept of cluster algebras [4] is particularly well suited to describing precisely this sort of system, and it is clear from the definition of cluster algebras that the two must be connected (this was also remarked in [10]). In fact,  $Y$ -systems, which are closely connected to  $Q$ -systems [18], have previously been studied using the cluster algebra formalism [6].

In this paper, we explain how to formulate this relation precisely in the case of simply-laced Lie algebras. The result is the description of the  $Q$ -systems as a subgraph of a cluster algebra tree which has particularly nice properties.

Section 2 is a short exposition of the background of the  $Q$ -system. In section 3, we recall some basic facts about cluster algebras, and show that the  $Q$ -system defines a particular case of a cluster algebra. We also discuss the relation of the polynomiality property, which can be proven using purely representation–theoretical arguments [11, 19], to the remarkable property of cluster algebras known as the Laurent phenomenon [5].

## 2. Background

### 2.1. The generalized Heisenberg spin chains and the Kirillov–Reshetikhin conjectures

Below we make reference to the Bethe equations, but since we will use only certain combinatorial constructions derived from these equations rather than the equations themselves, we refer the reader to the literature for further information on them [13, 20].

The generalized inhomogeneous Heisenberg model associated with the Yangian  $Y(\mathfrak{g})$  of a simple Lie algebra  $\mathfrak{g}$  was formulated in [12] for the classical Lie algebras. It can similarly be formulated for the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$ . This latter generalization corresponds to the deformation of the  $XXX$  model to the  $XXZ$  model.

Let  $R$  be either the rational  $R$ -matrix corresponding to the Yangian, or the trigonometric  $R$ -matrix, corresponding to the quantum affine algebra. The transfer matrix of the model is the trace over the auxiliary space  $V_0$  of a product of  $R$ -matrices:

$$T_{V_0}(z_1, \dots, z_N) = \text{Trace}_{V_0} R_{V_1, V_0}(z_1) \cdots R_{V_N, V_0}(z_N). \quad (2.1)$$

Here, the  $R$ -matrix  $R_{V, V'}$  is the intertwiner of certain special finite-dimensional representations of the Yangian  $Y(\mathfrak{g})$  or quantum affine algebra, which is obtained via fusion from the fundamental representations.

The inhomogeneity in the model is both in the spectral parameters  $z_i$  associated with each lattice site  $i$ , as well as in the representations  $V_i(z_i)$  at each site.

However, each of the representations  $V_i(z_i)$  are assumed to be of Kirillov–Reshetikhin (KR) type [12]. This type of representation can be defined, for example, by means of its Drinfeld polynomials, although the modules were originally defined in terms of fusion of  $R$ -matrices instead.

One way to define (and describe) Kirillov–Reshetikhin modules for  $Y(\mathfrak{g})$  for any Lie algebra  $\mathfrak{g}$  is to say that it is the smallest irreducible  $Y(\mathfrak{g})$ -module with  $\mathfrak{g}$ -highest weight which is proportional to one of the fundamental weights. Therefore, KR-modules are labeled by a triple,  $\alpha \in I_r$ , where  $I_r$  are the labels of the nodes in the Dynkin diagram,  $i \in \mathbb{Z}_+$  is

a non-negative integer and  $z \in \mathbb{C}^*$  is a complex number, corresponding to the localization parameter. The  $\mathfrak{g}$ -highest weight of such a module is  $i\omega_\alpha$ , where  $\omega_\alpha$  is the fundamental weight corresponding to the label  $\alpha$ , and the module can be denoted by  $V_{\alpha,i}(z)$ .

Kirillov and Reshetikhin were able to describe the structure of the spaces  $V_{\alpha,i}(z)$  as  $\mathfrak{g}$ -modules using the Bethe ansatz equations for such models. The results turned out to have deep combinatorial and representation–theoretical implications.

The Hilbert space of model (2.1) is

$$\mathcal{H} = \bigotimes_{i=1}^N V_i, \tag{2.2}$$

where  $N$  is the number of lattice sites. The *completeness hypothesis* is that the Bethe vectors provide a basis for the set of  $\mathfrak{g}$ -highest weight vectors in the Hilbert space, in the following sense.

Note that we always have  $\mathfrak{g} \subset Y(\mathfrak{g})$ , or  $U_q(\mathfrak{g}) \subset U_q(\widehat{\mathfrak{g}})$  in the case of the trigonometric  $R$ -matrix. Thus, the Hilbert space  $\mathcal{H}$  decomposes into a direct sum of finite-dimensional  $\mathfrak{g}$  or  $U_q(\mathfrak{g})$ -modules:

$$\mathcal{H} \simeq \bigoplus_{\mathfrak{g}\text{-mod } \lambda \in P^+} V(\lambda)^{\oplus m_{\lambda,\mathcal{H}}},$$

where  $V(\lambda)$  is the irreducible  $\mathfrak{g}$ -module (or  $U_q(\mathfrak{g})$ -module) characterized by the highest weight  $\lambda$ . Here,  $P^+$  is the set of dominant integral weights of  $\mathfrak{g}$ .

The multiplicity is the dimension of the space of  $\mathfrak{g}$ -linear homomorphisms,

$$m_{\lambda,\mathcal{H}} = \dim(\text{Hom}_{\mathfrak{g}}(\mathcal{H}, V(\lambda))). \tag{2.3}$$

The completeness conjecture is that the number of linearly independent Bethe vectors characterized by  $\lambda$  is equal to  $m_{\lambda,\mathcal{H}}$ .

Inasmuch as the dimensions of the representations, and hence the counting arguments, are concerned, there is no dependence on whether the  $R$ -matrix is rational or trigonometric (assuming that  $q$  is generic, in particular, that it is not a root of unity). There is no dependence on the choice of auxiliary space, since it does not influence the dimension of the Hilbert space.

For each  $\lambda$ , assuming the string hypothesis for the solutions to the Bethe equations<sup>1</sup>, the Bethe integers parametrize the Bethe vectors. These are sets of distinct integers, one set for each pair  $(\alpha, i)$  where  $\alpha \in I_r$  corresponds to one of the simple roots of the underlying algebra  $\mathfrak{g}$ , and  $i \in \mathbb{N}$  is the length of the ‘string’. These integers correspond to the string centers.

Given a set  $\{n_{\alpha,i} \in \mathbb{Z}_+ : \alpha \in I_r, i \in \mathbb{N}\}$ , choose a set of  $m_{\alpha,i}$  distinct integers are chosen from the interval  $[0, p_{\alpha,i}]$ , where

$$p_{\alpha,i} = \sum_j n_{\alpha,j} \min(i, j) - \sum_\beta \text{sgn}(C_{\alpha,\beta}) \sum_j \min(|C_{\alpha,\beta}|j, |C_{\beta,\alpha}|i) m_{\beta,j}, \quad (\alpha \in I_r, i \in \mathbb{N}). \tag{2.4}$$

These so-called *vacancy numbers* are determined from the large  $N$  analysis (where  $N$  is the number of sites) of the thermodynamic Bethe ansatz equations. The integers  $n_{\alpha,i}$  are parameters of the model—they parametrize the representations  $V_i(z_i)$  in the definition of the model and hence  $\mathcal{H}$ .

There are two restrictions on the choice of numbers  $\{m_{\alpha,i}\}$ . The first is clearly that the vacancy numbers should be non-negative,

$$p_{\alpha,i} \geq 0. \tag{2.5}$$

<sup>1</sup> Although the string hypothesis is known to fail in various scenarios, the counting algorithm of [13] turns out to give the correct number of states nonetheless.

The second keeps track of the sector  $\lambda$  to which the solutions belong:

$$\sum_i i n_{\alpha,i} - \sum_{\beta,i} i C_{\alpha,\beta} m_{\beta,i} = l_\alpha, \tag{2.6}$$

where the weight is  $\lambda = \sum_\alpha l_\alpha \omega_\alpha$  and  $\omega_\alpha$  are the fundamental weights of  $\mathfrak{g}$ . Here,  $C$  is the Cartan matrix of the Lie algebra  $\mathfrak{g}$ , with the convention that

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad \alpha_k \in \Pi.$$

The number of ways to choose  $m$  distinct integers from the interval  $[0, p]$  is the binomial coefficient:

$$\binom{m+p}{m} = \frac{(p+m)(p+m-1)\cdots(p+1)}{m!}. \tag{2.7}$$

Thus, for fixed  $\lambda$  and  $\mathbf{n} = (n_{\alpha,i})_{i \in \mathbb{N}; \alpha \in I}$ , the number of solutions to the Bethe equations is

$$M_{\lambda;\mathbf{n}} = \sum_{\substack{m_{\alpha,i} \geq 0 \\ p_{\alpha,i} \geq 0}} \prod_{\alpha} \binom{m_{\alpha,i} + p_{\alpha,i}}{m_{\alpha,i}}, \tag{2.8}$$

where the sum is taken over all non-negative integers  $\mathbf{m} = (m_{\alpha,i})$  such that equation (2.6) holds.

The completeness conjecture is therefore the following.

**Conjecture 2.1** [13]. *The dimension of the space of  $\mathfrak{g}$ -linear homomorphisms (2.3) is*

$$m_{\lambda;\mathcal{H}} = M_{\lambda,\mathbf{n}},$$

where  $\mathbf{n} = (n_{\alpha,i})$  and  $\mathbf{n}_{\alpha,i}$  is the number of Kirillov–Reshetikhin-type modules  $V_j$  in the tensor product (2.2) which have a  $\mathfrak{g}$ -highest  $i\omega_\alpha$ .

Until recently, a proof of this conjecture was available in various special cases, by proving a bijection between the set of ‘rigged configurations’ which enumerate the Bethe integers of the form described above, and crystal paths (see [13–15] for example).

However, a completely general proof is available for the following statement.

**Theorem 2.2** [11, 19]. *The dimension  $m_{\lambda;\mathcal{H}}$  of the space of homomorphisms is equal to the so-called  $N$ -sum [8, 13]:*

$$N_{\lambda;\mathbf{n}} = \sum_{m_{\alpha,i} \geq 0} \prod_{\alpha} \binom{m_{\alpha,i} + p_{\alpha,i}}{m_{\alpha,i}}, \tag{2.9}$$

where the summation is taken over all non-negative integers  $\mathbf{m}$  subject to the restriction (2.6), but not subject to the positivity condition (2.5).

Note that the binomial coefficient with definition (2.7) is perfectly well defined for the values of  $p < 0$  as long as  $m < -p$ . In fact, there is an identity,

$$\binom{m+p}{m} = (-1)^m \binom{-p-1}{m}.$$

Although at first glance, there is a little difference between the  $M$ -sum (2.8) and the  $N$ -sum (2.9), it is, in fact, a rather subtle combinatorial identity. The obvious difference is that the  $M$ -sum is a summation over manifestly non-negative terms, where in the  $N$ -sum, there are many negative terms, and many cancellations take place. However, the two sums are equal.

**Theorem 2.3** [3].

$$M_{\lambda;\mathbf{n}} = N_{\lambda;\mathbf{n}}. \tag{2.10}$$

This provides a proof of the combinatorial Kirillov–Reshetikhin conjecture 2.1 for all simple Lie algebras  $\mathfrak{g}$ .

2.2. *Q-systems*

Theorems 2.2 and 2.3 are, in fact, corollaries of a more basic fact, as was shown by [8], about the nature of the solutions of an associated *Q*-system.

First, let us describe the notion of polynomiality in the case that  $\mathfrak{g} = \mathfrak{sl}_2$ . In this case, Kirillov–Reshetikhin modules are in one-to-one correspondence with the irreducible finite-dimensional representations of  $\mathfrak{sl}_2$  and have the same dimension. That is, they are irreducible as  $\mathfrak{sl}_2$ -modules if we start with the Yangian module, or as a  $U_q(\mathfrak{sl}_2)$ -module if we refer to the quantum affine algebra.

Let  $Q_0 = 1$  be the character of the one-dimensional representation on which  $\mathfrak{sl}_2$  acts trivially, and  $Q_1$  be the character of the defining or fundamental representation of dimension two. For simplicity, define  $t = Q_1$ . The character of the  $(j + 1)$ -dimensional irreducible representation, with the highest weight  $j\omega_1$ , is then known to be polynomial—in fact, a Chebyshev polynomial of the second kind—in  $t$ . This is just a statement of the fact that any irreducible representation of  $\mathfrak{sl}_2$  is a polynomial representation, that is, it can be obtained from the image of the Young symmetrizer acting on the tensor product of the fundamental representation.

One can show that the Chebyshev polynomials  $Q_j$ , or, equivalently, the characters of the  $(j + 1)$ -dimensional irreducible representations of  $\mathfrak{sl}_2$ , satisfy a recursion relation

$$Q_{j+1}Q_{j-1} + 1 = Q_j^2.$$

This is the *Q*-system for  $\mathfrak{sl}_2$ .

It turns out that for any simple Lie algebra there is a similar system of recursion equations. For simply-laced simple Lie algebras, we write  $\beta \sim \alpha$  if the nodes  $\beta$  and  $\alpha$  are connected in the Dynkin diagram of  $\mathfrak{g}$ . Then the *Q*-system for a simply-laced Lie algebra  $\mathfrak{g}$  is

$$Q_{\alpha,k+1} = Q_{\alpha,k-1}^{-1} \left( Q_{\alpha,k}^2 - \prod_{\beta \sim \alpha} Q_{\beta,k} \right), \quad (k > 0), \quad Q_{\alpha,0} = 1, \quad Q_{\alpha,1} = t_\alpha. \tag{2.11}$$

There are also *Q*-systems for any other simple Lie algebra [8, 13] and for twisted affine algebras [9]. We do not discuss these systems in this paper.

We set the formal parameter  $t_\alpha$  to be the  $\mathfrak{g}$ -character of the ‘fundamental KR-module’  $Q_{\alpha,1}$ , corresponding to the highest weight  $\omega_\alpha$ . In general, this is not necessarily an irreducible  $\mathfrak{g}$ -module. For example, if  $\mathfrak{g} = D_4$ ,

$$Q_{2,1} = \text{ch}V(\omega_2) + \text{ch}V(0),$$

where  $V(0)$  is the trivial representation. These modules are also known as fundamental representations of Yangians (or minimal affinizations) and were studied in [1, 16].

Under fairly mild asymptotics conditions [8], it can be shown that the solutions of the *Q*-systems are characters of the KR-modules. This has been done for the special case above by Nakajima [19] and in complete generality by Hernandez [10].

In the case of  $\mathfrak{sl}_n$ , KR-modules are in fact the irreducible  $\mathfrak{sl}_n$  representations with ‘rectangular highest weights’ the form  $i\omega_\alpha$ .

In the case where  $\mathfrak{g}$  is not of type  $A$ , the characters in question are sometimes not irreducible as  $\mathfrak{g}$ -modules. However, they have a triangular decomposition in terms of irreducible characters with smaller highest weights, in the sense of partial ordering of highest weights. That is,

$$V_{\alpha,i}(z) \simeq V(i\omega_\alpha) \oplus \left( \bigoplus_{\lambda < i\omega_\alpha} V(\lambda)^{m_{\lambda,i\omega_\alpha}} \right). \tag{2.12}$$

The decomposition multiplicities are given by the Kirillov–Reshetikhin conjecture.

In [8] was shown that theorem 2.2 is a consequence of the fact that the characters of KR-modules solve the  $Q$ -system.

In proving theorem 2.3, an essential ingredient was this fact, and the following lemma, which follows entirely from representation–theoretical arguments and the theorem of [19] or more generally [11] (see also [17]).

**Lemma 2.4** [3]. *The solutions of the  $Q$ -system, for any Lie algebra, are a polynomial in the variables  $\{Q_{\alpha,1} | \alpha \in I_r\}$ .*

**Proof.** This follows from (2.12), which shows that the transition matrix between KR-modules and irreducible  $\mathfrak{g}$ -modules is unitriangular and hence invertible. Thus, any KR-module is in the Grothendieck ring generated by the fundamental and trivial representations of  $\mathfrak{g}$ . That is, the character  $Q_{\alpha,i}$  is a polynomial in the characters of the fundamental  $\mathfrak{g}$ -modules. Finally, using relation (2.12) again, it is seen to be a polynomial in  $Q_{\beta,1}$ .  $\square$

This statement can be rephrased as follows: the solutions of the  $Q$ -system are polynomials in the variables  $\{Q_{\alpha,1}\}_{\alpha \in I_r}$  in the limit  $Q_{\alpha,0} \rightarrow 1$ . In general, we do not know how to show this polynomiality property directly from the  $Q$ -system.

The proof of this statement relies entirely on the fact that the characters of KR-modules are solutions of the  $Q$ -system. However, the  $Q$ -system can be viewed as a dynamical system, or a recursion relation, independently of this fact. We show below that it can be expressed as a cluster algebra.

Seen in this light, the polynomiality property appears to be closely connected to the Laurent phenomenon [5] for cluster algebras (this was also recently suggested by Hernandez [10]). The rest of this paper is devoted to describing the cluster algebra structure of  $Q$ -systems, as a first step towards understanding this phenomenon. We will show in section 3.2 that the  $Q$ -system for simply-laced Lie algebras is, in fact, a quotient of a subgraph of a cluster algebra.

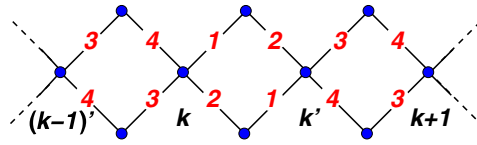
### 3. $Q$ -systems as cluster algebras

#### 3.1. Cluster algebras

For the definition of a cluster algebra, we refer the reader to the excellent summary in [7]. We recap this definition briefly here, specialized to the particularly simple case we consider in the context of  $Q$ -systems corresponding to simply-laced simple Lie algebras.

First, consider a regular  $n$ -ary tree  $\mathbb{T}_n$ . This is a tree with certain nodes, each node being connected to  $n$  other nodes via undirected, labeled edges. Each node is connected to edges labeled by the distinct labels  $1, \dots, n$ .

At each node there is a seed of the form  $(\mathbf{x}, B)$ . (More generally, there are also coefficients  $\mathbf{y}$  at each node, but in the case under consideration here, it is possible to take a normalized cluster algebra, with all coefficients as 1, so we do not consider the more general case of nontrivial coefficients.) Here,  $\mathbf{x}$  is an  $n$ -vector with entries  $x_i$  and  $B$  is a skew-symmetric  $n \times n$



**Figure 1.** The subgraph  $\mathbb{T}_g$  for  $sl_3$ , where the cluster variable  $\mathbf{x}$  is a 4-vector. The label  $i$  on an edge denotes the mutation  $\mu_i$  relating the two nodes. The special nodes  $k, k'$  and so forth are the nodes at the opposite ends of the squares.

integer matrix with entries  $B_{ij}$ . When we want to be explicit we will label the seed by the node to which it corresponds, e.g.  $(\mathbf{x}, B)_t$  is the seed at node  $t$ .

Nodes connected by an edge labeled  $k$  correspond to a mutation of the seed, or an evolution of the system in the direction  $k$ . This mutation is denoted by a map  $\mu_k : (\mathbf{x}, B) \mapsto (\mathbf{x}', B')$ . The map acts as follows on the seed:

$$\mu_k(x_i) = \begin{cases} x_i, & i \neq k; \\ x_k^{-1} \left( \prod_{j=1}^n x_j^{[B_{jk}]_+} + \prod_{j=1}^n x_j^{[-B_{jk}]_+} \right), & i = k. \end{cases} \quad (3.1)$$

$$\mu_k(B_{ij}) = \begin{cases} -B_{ij}, & \text{if } i = k \text{ or } j = k; \\ B_{ij} + \text{sign}(B_{ik})[B_{ik}B_{kj}]_+, & \text{otherwise.} \end{cases} \quad (3.2)$$

Here,  $[n]_+$  means the positive part of the integer  $n$ . It is equal to 0 if  $n \leq 0$ , and is equal to  $n$  if  $n > 0$ .

One can immediately check that  $\mu_i \circ \mu_i(\mathbf{x}, B) = (\mathbf{x}, B)$ , that is why the edges of the tree are not oriented.

A cluster pattern is an assignment of a labeled seed to each node  $t$  in  $\mathbb{T}_n$ , such that the seeds at connected nodes are related by the relevant seed mutations. A cluster algebra is the algebra in the cluster variables  $\mathbf{x}_t$  which are related by such mutations.

Thus, one has an  $n$ -ary tree  $\mathbb{T}_n$  and an evolution on it, connecting its nodes along the edges determined by the mutation matrix  $B$ . Starting from any node, one can compute the cluster variables  $\mathbf{x}$  and the mutation matrix  $B$  for any other node by traversing the tree.

A remarkable property of the mutations of a cluster algebra determined from such a matrix  $B$  is the *Laurent phenomenon*. This is a theorem which states that the cluster variables at any node are Laurent polynomials in the cluster variables  $\mathbf{x}$  at any other node [5]. This is a highly non-trivial result, because in general, mutations (3.1) give the cluster variables as a rational function in the cluster variables at another node.

For any given seed  $(\mathbf{x}, B)$ , one can ask about the structure of the evolution on the tree determined by such a seed. If we identify nodes corresponding to identical seeds, we obtain a quotient graph with a structure determined by the seed. For example, it is easy to see that  $\mu_i \circ \mu_j = \mu_j \circ \mu_i$  if  $B_{ij} = 0$ . Thus, one can consider the quotient graph where the nodes corresponding to such an equivalence relation are identified (see figure 1 for an example).

Below we formulate the  $Q$ -system as a quotient of a subgraph of a cluster algebra determined by a particularly simple matrix  $B$ .

### 3.2. The $Q$ -system as a cluster algebra

Let  $\mathfrak{g}$  be a simply-laced, simple Lie algebra of rank  $r$ , and let  $C$  be its Cartan matrix.



Consider the family of commutative variables  $\{Q_{\alpha,i} : \alpha \in I_r, i \in \mathbb{Z}\}$  defined by the recursion relation

$$Q_{\alpha,k+1} = Q_{\alpha,k-1}^{-1} \left( Q_{\alpha,k}^2 - \prod_{\beta \sim \alpha} Q_{\beta,k}^{-C_{\alpha,\beta}} \right). \tag{3.3}$$

With the initial conditions  $Q_{\alpha,0} = 1$  and  $Q_{\alpha,1} = t_\alpha$  (a formal variable), this is the  $Q$ -system (2.11) if one considers only  $k \geq 1$ .

In order to make contact with the usual definition of cluster algebras, it is useful to normalize the  $Q$ -system (3.3) to get rid of the minus signs. This is possible for the  $Q$ -system associated with any simple Lie algebra.

**Lemma 3.1.** *For each simple Lie algebra  $\mathfrak{g}$ , there exists a set of complex numbers  $\{\epsilon_\alpha\}_{\alpha \in I_r}$  such that the normalized cluster variables  $\tilde{Q}_{\alpha,k} = \epsilon_\alpha Q_{\alpha,k}$  satisfy the normalized system:*

$$\tilde{Q}_{\alpha,k+1} = \frac{\tilde{Q}_{\alpha,k}^2 + \prod_{\beta \sim \alpha} \tilde{Q}_{\beta,k}}{\tilde{Q}_{\alpha,k-1}}.$$

**Proof.** From the  $Q$ -system (3.3), we see that the normalization is a choice of  $\{\epsilon_\alpha\}_{\alpha \in I_r}$  such that

$$\prod_{\beta} \epsilon_\beta^{C_{\alpha,\beta}} = -1, \quad \forall \alpha. \tag{3.4}$$

Let  $\epsilon_\beta = e^{\mu_\beta}$  where  $\mu_\beta \in \mathbb{C}$ . Since  $C$  is invertible, (3.4) shows that it is sufficient to set  $\mu_\beta = i\pi \sum_{\alpha} C_{\beta,\alpha}^{-1}$ .  $\square$

**Remark 3.2.** This is true for the  $Q$ -system associated with any simple Lie algebra [8], not just the simply-laced case, following an almost identical proof to the one above. One simply notes that the total degree of all characters  $Q_{\beta,j'}$  appearing on the right-hand side of the  $Q$ -system for  $Q_{\alpha,j+1}$  is  $|C_{\alpha\beta}|$ , with  $\beta \sim \alpha$  in each case.

This choice of normalization is not particularly natural from the point of view of  $Q$ -systems in general, but makes the formulation as a cluster algebra simpler.

We claim that the  $Q$ -system corresponds to the evolution of a certain seed with a certain mutation matrix, defined on a subgraph of the  $n$ -ary tree  $\mathbb{T}_n$ , where  $n = 2r$  and  $r$  is the rank of  $\mathfrak{g}$ .

**Definition 3.3.** *Define the node labeled by  $k \in \mathbb{Z}$  as the node corresponding to the cluster variable  $\mathbf{x} = (x_1, \dots, x_{2r})$  given by*

$$x_\alpha = \tilde{Q}_{\alpha,2k}, \quad x_{\alpha+r} = \tilde{Q}_{\alpha,2k+1}, \quad (\alpha \leq r). \tag{3.5}$$

The mutation matrix  $B$  at the node  $k$  is defined as follows: for all  $\alpha, \beta \in I_r$ ,

$$B_{\alpha+r,\beta} = C_{\alpha\beta}, \quad B_{\alpha\beta} = B_{\alpha+r,\beta+r} = 0, \quad B_{\beta\alpha} = -B_{\alpha\beta}.$$

That is, if  $C$  is the Cartan matrix of a simply-laced Lie algebra, the matrix  $B$  is

$$B = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}, \tag{3.6}$$

which is skew-symmetric because  $C$  is symmetric in this case.

**Lemma 3.4.** *The mutations  $\mu_\alpha$  ( $\alpha \in I_r$ ), when applied to the seed  $\mathbf{x}$ , commute. That is*

$$\mu_\alpha \circ \mu_\beta(\mathbf{x}, B) = \mu_\beta \circ \mu_\alpha(\mathbf{x}, B), \quad \alpha, \beta \in I_r.$$

The same statement holds for the mutations  $\mu_{\alpha+r}$  with  $\alpha \in I_r$ :

$$\mu_{\alpha+r} \circ \mu_{\beta+r}(\mathbf{x}, B) = \mu_{\beta+r} \circ \mu_{\alpha+r}(\mathbf{x}, B), \quad \alpha, \beta \in I_r.$$

**Proof.** This is the result of the fact that the  $r \times r$  diagonal blocks of  $B$  vanish. In general, the mutations  $\mu_\alpha$  and  $\mu_\beta$  commute whenever  $B_{\alpha,\beta} = 0$ .  $\square$

In fact, some of the mutations from the two sets  $\{1, \dots, r\}$  and  $\{r + 1, \dots, 2r\}$  commute, depending on the structure of the Cartan matrix. However, we do not need this structure in order to describe the cluster algebra corresponding to the  $Q$ -system.

We now describe a particularly simple subgraph of the tree  $\mathbb{T}_{2r}$ , whose nodes correspond to cluster variables consisting only of members of the family  $\{Q_{\alpha,i}\}$ , and whose edges correspond only to mutations describing one of the recursion relations in the  $Q$ -system (3.3).

In fact we will consider a quotient of this graph, obtained by identifying nodes of the subgraph which have the same seed. In particular, we take a quotient of the graph by the relations  $[\mu_\alpha, \mu_\beta] = 0$  if  $\alpha, \beta \in I_r$  or  $\alpha - r, \beta - r \in I_r$ .

Starting from the node  $k$ , consider any subset of  $I_r$ , and traverse the graph along the edges labeled by these elements. Since the mutations corresponding to these elements commute, they can be taken in any order. Since  $\mu_\alpha^2 = 1$ , they can be assumed to be distinct.

**Definition 3.5.** The node  $k'$  ( $k \in \mathbb{N}$ ) is the node reached from the node  $k$  by traversing edges labeled by all elements of  $I_r$ .

This node is unique and independent of the order of the mutations in this subset.

**Lemma 3.6.** The seed corresponding to the node  $k'$  is  $(\mathbf{x}', B') = \mu_r \circ \dots \circ \mu_1(\mathbf{x}, B)$ , where

$$\mathbf{x}' = \begin{pmatrix} \tilde{Q}_{1,2k+2} \\ \vdots \\ \tilde{Q}_{r,2k+2} \\ \tilde{Q}_{1,2k+1} \\ \vdots \\ \tilde{Q}_{r,2k+1} \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}.$$

**Proof.** The statement about the cluster variable  $\mathbf{x}$  follows from the definition of the  $Q$ -system. Note that  $\mu_\alpha(x_\beta) = x_\beta$  if  $\beta \neq \alpha$  and therefore  $x'_\beta$  corresponding to the node  $k'$  has  $x'_\alpha = \mu_\alpha(x_\alpha) = \tilde{Q}_{\alpha,2k}^{-1}(\tilde{Q}_{\alpha,2k+1}^2 + \prod_{\beta \sim \alpha} \tilde{Q}_{\beta,2k+1}) = \tilde{Q}_{\alpha,2k+2}$  where  $\alpha \in I_r$ .

Next, we note that if  $k, l \neq i$  with  $i, l \in I_r$ , then  $\mu_i(B_{k,l}) = B_{k,l}$ . This is because  $B_{il} = 0$ , whereas  $\mu_i(B_{kl}) = B_{kl} + \text{sgn}(B_{ki})[B_{ki} B_{il}]_+ = B_{kl} + 0$ .

Moreover,  $\mu_i(B_{ki}) = -B_{ki}$  for  $1 \leq k \leq 2r$ . Since the mutation  $\mu_i$  maintains skew-symmetry so  $\mu_i(B_{lk}) = B_{lk}$  if  $l \in I_r$  and  $k, l \neq i$ , and  $\mu_i(B_{ik}) = -B_{ik}$  if  $k \leq r$ .

This shows that

$$\mu_r \circ \dots \circ \mu_1(B) = \begin{pmatrix} 0 & C \\ -C & A \end{pmatrix},$$

where  $A$  is some matrix to be determined. The following lemma shows that the matrix  $A$  is 0 if we start from the matrix  $B$  as in (3.6).  $\square$

**Lemma 3.7.** Define

$$B = \begin{pmatrix} 0 & -C \\ C & A \end{pmatrix},$$

where  $C$  is any Cartan matrix and  $A$  is any matrix. Then  $B' := \mu_1 \circ \dots \circ \mu_r(B)$  has the form

$$B' = \begin{pmatrix} 0 & C \\ -C & A \end{pmatrix}.$$

**Proof.** According to lemma 3.6, the transformation  $\mu_1 \circ \dots \circ \mu_r$  changes the sign of the first  $r$  rows and columns of  $B$ , because  $B_{kl} = 0$  if  $k, l \leq r$ .

We must therefore check that the transformation leaves the matrix  $A$  invariant. That is, we must show that for  $j, k \in [1, r]$

$$\begin{aligned} 0 &= B'_{r+j, r+k} - B_{r+j, r+k} \\ &= \sum_{i=1}^r \operatorname{sgn}(B_{r+j, i}) [B_{r+j, i} B_{i, r+k}]_+ \\ &= \sum_{i=1}^r \operatorname{sgn}(C_{j, i}) [-C_{j, i} C_{i, k}]_+ \\ &= \operatorname{sgn}(C_{jj}) [-C_{jj} C_{jk}]_+ + \operatorname{sgn}(C_{jk}) [-C_{jk} C_{kk}]_+ + \sum_{i \neq j, k} \operatorname{sgn}(C_{ji}) [-C_{ji} C_{ik}]_+ \\ &= 2(1 - \delta_{jk}) |C_{jk}| - 2(1 - \delta_{jk}) |C_{jk}| + 0 = 0. \end{aligned}$$

In the first two terms, we used the fact that the diagonal elements of  $C$  are 2 and the off-diagonal elements are negative. In the last term, we used the fact that both  $C_{ji}$  and  $C_{ik}$  are off-diagonal and hence their product is positive.  $\square$

Graphically, the result of applying the mutations  $\mu_\alpha$  ( $\alpha \in I_r$ ) sequentially starting with the node  $k$  and identifying nodes corresponding to different orderings of distinct mutations is a hypercube of dimension  $r$ , with two special vertices,  $k$  and its mirror image  $k'$ . After application of the  $r$  distinct mutations  $\{\mu_1, \dots, \mu_r\}$  to the node  $k$  in any order, one ends up at the node which we labeled  $k'$ , which is at the opposite end of the hypercube. See, for example, figure 1 for the case of  $\mathfrak{sl}_3$ .

Similarly, starting from the node  $k \in \mathbb{N}$ , we can consider the subtree of  $\mathbb{T}_{2r}$  generated by edges labeled the set  $\{\alpha + r\}$  with  $\alpha \in I_r$ . The corresponding mutations also commute among themselves. We again take the quotient by identifying nodes with the same seed. The result is another hypercube, which is also given in figure 1.

An almost identical analysis for this sequence of mutations holds as for the mutations  $\mu_\alpha$  with  $\alpha \in I_r$ . Applying the sequence of mutations  $\mu_{2r} \circ \dots \circ \mu_{r+1}$  to the seed  $(\mathbf{x}, B)_k$  we arrive at the node which we label  $(k - 1)'$ . We have the following lemma.

**Lemma 3.8.** *The seed corresponding to the node  $(k - 1)'$  is*

$$\mathbf{x}' = \begin{pmatrix} \tilde{Q}_{1,2k} \\ \vdots \\ \tilde{Q}_{r,2k} \\ \tilde{Q}_{1,2k-1} \\ \vdots \\ \tilde{Q}_{r,2k-1} \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}.$$

The proof is virtually identical to the one above, except that we are moving ‘backwards’ in the  $Q$ -system, defining  $Q_{\alpha, 2k-1}$  from  $Q_{\alpha, 2k+1}$  and  $\{Q_{\beta, 2k}\}_\beta$  using the  $Q$ -system.

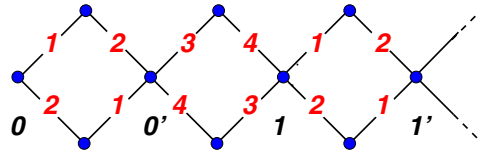


Figure 2. The subgraph  $\overline{\mathbb{T}}_{\mathfrak{sl}_3}$ .

From the node  $k'$ , we may now proceed acting with the mutations  $\{\mu_{r+1}, \dots, \mu_{2r}\}$ , taken to be distinct and in any order. The result of applying all  $r$  mutations of this type is the unique node  $k + 1$ , and a similar analysis shows that the seed corresponding to the node  $k + 1$  has the form  $(\mathbf{x}, B)_{k+1}$ , where  $B$  is the same as in equation (3.6) and  $\mathbf{x}$  has the same form as  $\mathbf{x}$  at  $k$ , but with  $k + 1$  substituted for  $k$ .

In general, we refer to acting ‘to the left’ and ‘to the right’, depending on whether we are starting from node  $k$  or  $k'$ . Acting to the right from a node  $k$  means acting with mutations  $\mu_i$  with  $i \leq r$ , and acting to the left means acting with  $\mu_i$  with  $i > r$ . Acting to the right on a node  $k'$  means acting with mutations  $\mu_i$  with  $i > r$ , and acting to the left on node  $k'$  means acting with mutations  $\mu_i$  with  $i \leq r$ .

We can continue to construct this graph in both directions, encountering only the two types of nodes corresponding to each integer  $k$  after applying the relevant set of  $r$  mutations.

Thus, we can define the graphs corresponding to the evolution of the  $Q$ -systems (3.3) and (2.11).

**Definition 3.9.** The graph  $\mathbb{T}_{\mathfrak{g}}$  is obtained from  $\mathbb{T}_{2r}$  by starting with any node  $k \in \mathbb{Z}$ , with the matrix  $B$  as in (3.6), and considering edges corresponding to the distinct mutations,  $1, \dots, r$ , taken in any order, identifying nodes with the same seeds.

The node reached after acting once with each mutation in this set is the node  $k'$ . The graph is extended to the right by then acting with the mutations  $\mu_{r+1}, \dots, \mu_{2r}$  to reach the node  $k + 1$ , and so forth.

The graph is extended from node  $k$  to the left by acting with the mutations  $r + 1, \dots, 2r$ , to reach node  $(k - 1)'$ . The graph is extended to the left from this node by considering the evolutions  $\mu_1, \dots, \mu_r$  to reach node  $(k - 1)$ , etc.

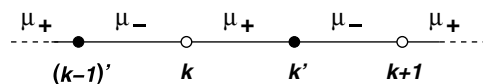
This graph as a chain of hypercubes, infinite in both directions, as in figure 1. In contrast, the graph corresponding to the original  $Q$ -system (2.11) is semi-infinite, with the cutoff achieved by setting initial conditions at  $k = 0$ .

**Definition 3.10.** The graph  $\overline{\mathbb{T}}_{\mathfrak{g}} \subset \mathbb{T}_{\mathfrak{g}}$  corresponding to the  $Q$ -system (2.11) is the subgraph of  $\mathbb{T}_{\mathfrak{g}}$  with a specialized seed at  $k = 0$ , where we set  $Q_{\alpha,0} = 1$ . The subgraph consists of the node  $k = 0$  and the nodes to its ‘right’. That is, first by the set of mutations  $\mu_1, \dots, \mu_r$ , then the mutations  $\mu_{r+1}, \dots, \mu_{2r}$ , and alternating these.

This can be considered as a semi-infinite chain of mutations. See figure 2 for the example of  $\mathfrak{sl}_3$ .

The initial condition at  $k = 0$  implies, in particular, that  $Q_{\alpha,-1} = 0$ .

In fact, all cluster variables  $\{\tilde{Q}_{\alpha,i}\}_{i \in \mathbb{Z}}$  are found by considering just the nodes  $k$  with  $k \in \mathbb{Z}$ , and ignoring the fine structure which is given by the Cartan matrix  $C$  (in fact, by its rank). The situation is similar to the ‘bipartite belt’ considered by Fomin and Zelevinsky in



**Figure 3.** The one-dimensional chain containing all cluster variables.

the context of  $Y$ -systems (which are of course closely connected to  $Q$ -systems). One may define the compound mutations

$$\mu_+ = \mu_r \circ \cdots \circ \mu_1, \quad \mu_- = \mu_{2r} \circ \cdots \circ \mu_{r+1}$$

and consider the chain, obtained by acting on the seed at node  $k$  with  $\mu_+$  and  $\mu_-$  alternatively. We have  $(\mu_+)^2 = (\mu_-)^2 = 1$ . Acting on the seed at node  $k$  with  $\mu_+$  brings us to node  $k'$ , and acting with  $\mu_-$  brings us to node  $(k - 1)'$ . Similarly, acting on node  $k'$  with  $\mu_-$  brings us to node  $k + 1$ . The result is a one-dimensional chain with two types of nodes, see figure 3. All cluster variables are contained in the unprimed nodes  $k \in \mathbb{Z}$ , or in the primed nodes  $k'$ .

The cluster variables which give the  $Q$ -system are contained in the subgraph with  $k \geq 0$ , with the cluster variables at  $k = 0$  given by  $Q_{\alpha,0} = 1$  and  $Q_{\alpha,1} = t_\alpha$ . This is a singular point, because the evolution  $\mu_{\alpha+r}$  with  $\alpha \in I_r$  is not invertible at this point, since  $\mu_{\alpha+r}$  has a non-trivial kernel when acting on  $(\mathbf{x})_0$ .

**Remark 3.11.** It is actually possible to make sense of the  $Q$ -system in the limit  $Q_{\alpha,0} \rightarrow 1$ , thus obtaining an infinite chain (rather than a semi-infinite one), with cluster variables at points  $k < 0$  which are 0 at a finite number of nodes, and are related by a sign to cluster variables at nodes with  $k > 0$ . We will give details of this extension in a future publication.

### 3.3. Strong Laurent phenomenon

One of the most remarkable properties of cluster algebras, given the rational nature of the mutations, is the *Laurent phenomenon* [5]. This is a theorem which states that any cluster variable is a Laurent polynomial in the other cluster variables, with coefficients in the group ring  $\mathbb{Z}\mathbb{P}$ , where  $\mathbb{P}$  is the coefficient ring (the integers in our case).

This is to be compared with the polynomiality property, for the  $Q$ -system (2.11) which is explained in lemma 2.4. Thus, a corollary of lemma 2.4 is the following theorem.

**Theorem 3.12.** *For the cluster algebra corresponding to the  $Q$ -system above, in the limit  $Q_{\alpha,0} \rightarrow 1$  (that is,  $\tilde{Q}_{\alpha,0}$  the appropriate root of unity, see lemma 3.1), the cluster variables at any node  $k > 0$  are polynomials in the cluster variables of the node  $k = 0$ .*

In fact, one can show that the same statement holds for the cluster variables at  $k < 0$ , corresponding to the extended  $Q$ -system, see remark 3.11. Most intriguingly, it appears that the phenomenon generalizes to other ‘branches’ of the tree  $\mathbb{T}_n$ , although this is certainly less than obvious from the  $Q$ -system!

This theorem is, in some sense, a stronger statement than the Laurent phenomenon, because we have no negative powers of  $Q_{\alpha,1}$  in  $Q_{\beta,k}$ . However, it says nothing about the dependence on other cluster variables at other nodes.

We note that it is precisely the Laurent phenomenon theorem that allows us to prove that the extension of the  $Q$ -system (2.11) to negative values of  $k$  is well defined in the limit  $Q_{\alpha,0} \rightarrow 1$ , because there can be no pole in the cluster variables at this point.

#### 4. Conclusions

In this paper, we have shown that it is possible to formulate the  $Q$ -system for simply-laced Lie algebras as a cluster algebra, and discussed the related graph structure. This gives a strong constraint on the cluster variables in the limit where  $Q_{\alpha,0} \rightarrow 1$ , according to theorem 3.12. This leaves many open problems.

The graph structure is more interesting, and includes more information about the Cartan matrix  $C$  (other than just its rank) if we allow all possible evolutions of the  $Q$ -system from any node in  $\mathbb{T}_{2r}$ . The simplest case where this comes into play is when the rank is 3 or greater. We have not discussed this structure in these notes.

The formulation of  $Q$ -systems for the non-simply-laced cases is more complicated and the definitions must be generalized to allow for such systems.

In general, it is more natural to consider systems with coefficients, allowing us to consider the un-normalized  $Q$ -systems, which contain minus signs.

Moreover, we have defined in [3] a deformed  $Q$ -system which contains more free parameters. We do not yet know whether these can be considered as cluster algebras, but they are interesting as they have an alternative formulation for their evolution involving substitution rather than a rational mutation map.

These and other extensions will be addressed in a future publication.

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